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A COMMON FIXED POINT THEOREM IN HILBERT- SPACE

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ABSTRACT

In this paper we obtain the fixed point theorem for eight continuous random operators defined on a non empty closed subset of a separable Hilbert-Space.

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KEYWORDS: Separable Hilbert Space, Random Operators, Common Fixed Point, Rational Inequality

1. INTRODUCTION

We construct a sequence of a Separable function in this paper and considering its convergence to a common unique Fixed Point [6][8] of eight continuous Random Operator defined on a non empty Closed subset of a Separable Hilbert Space.

Here we denote (P_1,P_2) as Measurable Space consisting of Sets P_1 & P_2, P_2 is subset of P_1, H stands for Separable Hilbert – Space and C is a non empty closed subset of H.

2. MAIN THEOREM

Let C be a non empty closed subset of a Separable Hilbert –Space H. Let P, Q, R, S, W, X, Y, and Z be eight continuous Random Operators defined on C such that

for $t \epsilon P_1$, P(t), Q(t), R(T), S(T), W(t), X(t), Y(t), Z(t)

$C \to C$ satisfy

If $PX = XP, WQ = QW, RZ = ZR, YS = SY, P(H) \subseteq W(H), Q(H) \subseteq X(H), R(H) \subseteq Y(H),$

and $S(H) \subseteq Z(H)$ ... (1)

And

$$\|Px - Sy\|^2 \leq \frac{\alpha_1 \|Zx - Px\|^2 [\|Wy - Sy\|^2 + \|Px - Wy\|^2]}{\|Zx - Wy\|^2 + \|Px - Wy\|^2}$$

$$+\frac{\alpha_2 \|Px - Wy\|^2 \left[\|Zx - Px\|^2 + \|Wy - Sy\|^2\right]}{\|Zx - Wy\|^2 + \|Px - Wy\|^2}$$

$$+\frac{\alpha_3 \|Zx - Px\|^2 \|Wy - Sy\|^2}{\|Zx - Wy\|^2}$$

$$+\frac{\alpha_4 \|Zx - Wy\|^2 \left[\|Zx - Px\|^2 + \|Wy - Sy\|^2\right]}{1 + \|Px - Wy\|^2}$$

$$+\alpha_5 \|Zx - Wy\|^2 \quad (2)$$

Proof: Let the function $g_0 : P_1 \to C$ be arbitrary separable function. From (1). ∃ a function $g_1 : P_1 \to C$ such that $W(t, g_1(t)) = P(t, g_0(t))\ for\ t \epsilon P_1$ and for this function $g_1 : P_1 \to C$, we can choose another function $g_2 : P_1 \to C$ such that $X(t, g_2(t)) = Q(t, g_1(t))\ for\ t\epsilon P_1$ and for this function $g_2 : P_1 \to C$ we can choose another function $g_3 : P_1 \to C$ such that $Y(t, g_3(t)) = R(t, g_2(t))$ and again for this function $g_3 : P_1 \to C$ we can choose another function $g_4 : P_1 \to C$ such that $Z(t, g_4(t)) = S(t, g_3(t))$.

By the induction method we can define a sequence of functions $for\ t\epsilon P_1$ and $\{Y_n(t)\}$ such that

$$Y_{2n}(t) = W\big(t, g_{2n+1}(t)\big) = P\big(t, g_{2n}(t)\big) \quad (3)$$

$$Y_{2n+1}(t) = X\big(t, g_{2n+2}(t)\big) = Q\big(t, g_{2n+1}(t)\big) \quad (4)$$

$$Y_{2n+2}(t) = Y\big(t, g_{2n+3}(t)\big) = R\big(t, g_{2n+2}(t)\big) \quad (5)$$

$$Y_{2n+3}(t) = Z\big(t, g_{2n+4}(t)\big) = S\big(t, g_{2n+3}(t)\big) \quad (6)$$

$for\ t\epsilon P_1\ and\ n = 0,1,2,3 \ldots..$

From (1) and (2) we have $for\ t\epsilon P_1$

$$\|Y_{2n}(t) - Y_{2n+3}(t)\|^2 = \|P(t, g_{2n}(t)) - S(t, g_{2n+3}(t))\|^2$$

$$\leq \frac{\alpha_1 \|Z(t,g_{2n+4}(t)) - P(t,g_{2n}(t))\|^2 [\|W(t,g_{2n+1}(t)) - S(t,g_{2n+3}(t))\|^2 + \|P(t,g_{2n}(t)) - W(t,g_{2n+1}(t))\|^2]}{\|Z(t,g_{2n+4}(t)) - W(t,g_{2n+1}(t))\|^2 + \|P(t,g_{2n}(t)) - W(t,g_{2n+1}(t))\|^2}$$

$$+\frac{\alpha_2 \| P(t,g_{2n}(t)) - W(t,g_{2n+1}(t))\|^2 [\|Z(t,g_{2n+4}(t)) - P(t,g_{2n}(t))\|^2 + \|W(t,g_{2n+1}(t)) - S(t,g_{2n+3}(t))\|^2]}{\|Z(t,g_{2n+4}(t)) - W(t,g_{2n+1}(t))\|^2 + \|P(t,g_{2n}(t)) - W(t,g_{2n+1}(t))\|^2}$$

$$+\frac{\alpha_3 \|Z(t,g_{2n+4}(t)) - P(t,g_{2n}(t))\|^2 \|W(t,g_{2n+1}(t)) - S(t,g_{2n+3}(t))\|^2}{\|Z(t,g_{2n+4}(t)) - W(t,g_{2n+1}(t))\|^2}$$

$$+\frac{\alpha_4 \|Z(t,g_{2n+4}(t)) - W(t,g_{2n+1}(t))\|^2 [\|Z(t,g_{2n+4}(t)) - P(t,g_{2n}(t))\|^2 + \|W(t,g_{2n+1}(t)) - S(t,g_{2n+3}(t))\|^2]}{1 + \|P(t,g_{2n}(t)) - W(t,g_{2n+1}(t))\|^2}$$

$$+\alpha_5 \|Z\big(t, g_{2n+4}(t)\big) - W\big(t, g_{2n+1}(t)\big)\|^2$$

$$\leq \frac{\alpha_1 \|Y_{2n+1}(t) - Y_{2n}(t)\|^2 \left[\|Y_{2n}(t) - Y_{2n+3}(t)\|^2 + \|Y_{2n}(t) - Y_{2n}(t)\|^2\right]}{\|Y_{2n+1}(t) - Y_{2n}(t)\|^2 + \|Y_{2n}(t) - Y_{2n}(t)\|^2}$$

$$+\frac{\alpha_2 \|Y_{2n}(t) - Y_{2n}(t)\|^2 [\|Y_{2n+1}(t) - Y_{2n}(t)\|^2 + \|Y_{2n}(t) - Y_{2n}(t)\|^2]}{\|Y_{2n+1}(t) - Y_{2n}(t)\|^2 + \|Y_{2n}(t) - Y_{2n}(t)\|^2}$$

$$+\frac{\alpha_3 \|Y_{2n+1}(t) - Y_{2n}(t)\|^2 \|Y_{2n}(t) - Y_{2n+3}(t)\|^2}{\|Y_{2n+1}(t) - Y_{2n}(t)\|^2}$$

$$+\frac{\alpha_4 \|Y_{2n+1}(t) - Y_{2n}(t)\|^2 [\|Y_{2n+1}(t) - Y_{2n}(t)\|^2 + \|Y_{2n}(t) - Y_{2n+3}(t)\|^2]}{1 + \|Y_{2n}(t) - Y_{2n}(t)\|^2}$$

$$+\alpha_5 \|Y_{2n+1}(t) - Y_{2n}(t)\|^2$$

$$\leq \alpha_1 \|Y_{2n}(t) - Y_{2n+3}(t)\|^2$$

$$+\alpha_2 [0]$$

$$+\alpha_3 \|Y_{2n}(t) - Y_{2n+3}(t)\|^2$$

$+\alpha_4\|Y_{2n}(t) - Y_{2n+3}(t)\|^2$

$+\alpha_4\|Y_{2n+1}(t) - Y_{2n}(t)\|^2$

$+\alpha_5\|Y_{2n+1}(t) - Y_{2n}(t)\|^2$

$\Rightarrow \|Y_{2n}(t) - Y_{2n+3}(t)\|^2$

$\leq (\alpha_1 + \alpha_3 + \alpha_4)\|Y_{2n}(t) - Y_{2n+3}(t)\|^2 + (\alpha_4 + \alpha_5)\|Y_{2n+1}(t) - Y_{2n}(t)\|^2$

$\Rightarrow \|Y_{2n}(t) - Y_{2n+3}(t)\|^2 [1 - (\alpha_1 + \alpha_3 + \alpha_4)] \leq (\alpha_4 + \alpha_5)\|Y_{2n+1}(t) - Y_{2n}(t)\|^2$

$\Rightarrow \|Y_{2n}(t) - Y_{2n+3}(t)\|^2 \leq \frac{(\alpha_4+\alpha_5)}{[1-(\alpha_1+\alpha_3+\alpha_4)]}\|Y_{2n+1}(t) - Y_{2n}(t)\|^2$

$\Rightarrow \|Y_{2n}(t) - Y_{2n+3}(t)\| \leq [\frac{(\alpha_4+\alpha_5)}{[1-(\alpha_1+\alpha_3+\alpha_4)]}]^{\frac{1}{2}}\|Y_{2n+1}(t) - Y_{2n}(t)\|$

Taking Q=$[\frac{(\alpha_4+\alpha_5)}{[1-(\alpha_1+\alpha_3+\alpha_4)]}]^{\frac{1}{2}}$

$\Rightarrow\|Y_{2n}(t) - Y_{2n+3}(t)\| \leq Q\,\|Y_{2n+1}(t) - Y_{2n}(t)\|$

Replacing 2n by n

$\Rightarrow \|Y_n(t) - Y_{n+3}(t)\| \leq Q\,\|Y_{n+1}(t) - Y_n(t)\|$

On further reducing

$\Rightarrow\|Y_n(t) - Y_{n+3}(t)\| \leq Q^{n+2}\,\|Y_1(t) - Y_0(t)\|$

$for\ t\epsilon P_1$

Now we shall prove that $for\ t\epsilon P_1$ $\{Y_n(t)\}$ is a Cauchy Sequence.

For this every positive integer we have

$\Rightarrow\|Y_n(t) - Y_{n+k}(t)\| = \|Y_n(t) - Y_{n+1}(t) + Y_{n+1}(t) - Y_{n+2}(t) + \ldots\ldots\ldots Y_{n+k-1}(t) - Y_{n+k}(t)\|$

$\leq [Q^{n+2} + Q^{n+1} + Q^n + \ldots Q^{n+k-1}]\,\|Y_1(t) - Y_0(t)\|$

$\leq [1 + Q + Q^2 + \ldots Q^{k-1}]\,Q^{n+2}\,\|Y_1(t) - Y_0(t)\|$

$\leq \frac{Q^n}{1-Q}\|Y_1(t) - Y_0(t)\|$

$\Rightarrow\|Y_n(t) - Y_{n+k}(t)\| \to 0\ as\ n \to \infty\ t\ \epsilon P_1$ .... (5)

Hence from equation it follows that for $t\epsilon P_1$ $\{Y_n(t)\}$ is a Cauchy sequence & hence is Convergent in closed subset C of a Hilbert Space H.

For $t\epsilon P_1$ let $\{Y_n(t)\} \to y(t)\}\ as\ n \to \infty$

Again as closeness of C given that g is a function from C to C.

And Consequently the sub sequence $P(t, g_{2n}(t)), Q(t, g_{2n+1}(t))$, $R(t, g_{2n+2}(t))$,

$S(t, g_{2n+3}(t))$, $W(t, g_{2n+1}(t)), X(t, g_{2n+2}(t))$, $Y(t, g_{2n+3}(t))$, $Z(t, g_{2n+4}(t))$, of $\{Y_n(t)\}$ for $t\epsilon P_1$ also converges to the $y(t)$

Continuity of P, Q, R, S, W, X, Y, and Z,

$P[t, X(t, g_n(t))] \to P(t, y(t))$

$X[t, P(t, g_n(t))] \to X(t, y(t))$

$Q[t, W(t, g_n(t))] \to Q(t, y(t))$

$W[t, Q(t, g_n(t))] \to W(t, y(t))$

$R[t, Z(t, g_n(t))] \to R(t, y(t))$

$Z[t, R(t, g_n(t))] \to Z(t, y(t))$

$Y[t, S(t, g_n(t))] \to Y(t, y(t))$ &

$S[t, Y(t, g_n(t))] \to S(t, y(t))$

And $P(t, y(t)) = X(t, y(t))$

$Q(t, y(t)) = W(t, y(t))$

$R(t, y(t)) = Z(t, y(t))$

$Y(t, y(t)) = S(t, y(t))$ for $t \epsilon P_1$

We have existence of a fixed point for $t \epsilon P_1$.

UNIQUENESS

Let $h: P_1 \to C$ be another fixed point common to P, Q, R, S, W, X ,Y and Z that is for $t \epsilon P_1$

$$\|g(t) - h(t)\|^2 = \|P(t, g(t)) - S(t, h(t))\|^2$$

$$\leq \frac{\alpha_1 \|Z(t,g(t)) - P(t,g(t))\|^2 [\|W(t,h(t)) - S(t,h(t))\|^2 + \|P(t,g(t)) - W(t,h(t))\|^2]}{\|Z(t,g(t)) - W(t,h(t))\|^2 + \|P(t,g(t)) - W(t,h(t))\|^2}$$

$$+ \frac{\alpha_2 \|P(t,g(t)) - W(t,h(t))\|^2 [\|Z(t,g(t)) - P(t,g(t))\|^2 + \|W(t,h(t)) - S(t,h(t))\|^2]}{\|Z(t,g(t)) - W(t,h(t))\|^2 + \|P(t,g(t)) - W(t,h(t))\|^2}$$

$$+ \frac{\alpha_3 \|Z(t,g(t)) - P(t,g(t))\|^2 \|W(t,h(t)) - S(t,h(t))\|^2}{\|Z(t,g(t)) - W(t,h(t))\|^2}$$

$$+ \frac{\alpha_4 \|Z(t,g(t)) - P(t,g(t))\|^2 + \|W(t,h(t)) - S(t,h(t))\|^2}{[1 + \|P(t,g(t)) - W(t,h(t))\|^2]}$$

$$+ \alpha_5 \|Z(t, g(t)) - (t, h(t))\|^2$$

$$\|g(t) - h(t)\|^2 \leq \alpha_5 \|g(t) - h(t)\|^2$$

$$(1 - \alpha_5) \|g(t) - h(t)\|^2 \leq 0 \; where \; \alpha_5 < 1/2$$

$g(t) = h(t)$ for $t \epsilon P_1$

This completes the proof of the theorem.

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